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To cite this article: Suratna Das *et al* JCAP06(2015)001

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Revisiting a pre-inflationary radiation era and its effect on the CMB power spectrum

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Abstract. We revisit the scenario where inflation is preceded by a radiation era by considering that the inflaton too could have been in thermal equilibrium early in the radiation era. Hence we take into account not only the effect of a pre-inflationary era on the inflaton mode functions but also that of a frozen thermal distribution of inflaton quanta. We initially discuss in detail the issues relevant to our scenario of a pre-inflationary radiation dominated era and then obtain the scalar power spectrum for this scenario. We find that the power spectrum is free from infrared divergences. We then use the WMAP and Planck data to determine the constraints on the inflaton comoving 'temperature' and on the duration of inflation. We find that the best fit value of the duration of inflation is less than 1 e-folding more than what is required to solve cosmological problems, while only an upper bound on the inflaton temperature can be obtained.

Contents

1 Introduction

The inflationary paradigm [1–5] successfully explains not only the spatial flatness, isotropy and homogeneity of our observed universe, but also the origin of the density fluctuations in the early universe which give rise to the large scale structure we observe today. Inflation, a period of quasi-exponential expansion of our universe, stretches (quantum) scalar fluctuations beyond the horizon thereafter freezing their amplitudes. These fluctuations may then be treated as classical, and are the source of the gravitational instabilities which later form the large scale structure of the universe. Though the existing observational evidence seems to support the inflationary paradigm, what happened before inflation is completely unknown. Could inflation be preceded by a radiation dominated era?

The possible consequences of a pre-inflationary radiation era have been studied earlier. It is well known in the literature [6–13] that the presence of a pre-inflationary radiation era, where one has 'just-enough' inflation, lowers the quadrupole moment of the CMB temperature anisotropy spectrum.[1] Lack of power in the CMB quadrupole is in accordance with observations like COBE [42], WMAP [43] and PLANCK [44] despite the issue of cosmic variance [45]. The transition from a pre-inflationary radiation era to a quasi-exponential

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[1]Other attempts to explain the low power at low CMB multipoles involve non-trivial topologies of the universe [14–22], bouncing cosmologies [23, 24], various inflationary scenarios (e.g. hybrid models of inflation [25], multi-field inflation [26], inflation which takes place in two stages [27–29] just enough inflation [30, 31] which could take place in modified gravity theories [32] or preceded by a fast roll phase [33]), etc. There have also been attempts at providing various other explanations such as non-primordial causes [34–38] and those based on systematic effects [39, 40] as well as attempts to relate this low power to other anomalies in the CMB (see e.g. [41]).

inflationary era, and its effect on the inflaton mode functions, has been studied (i) when the transition is instantaneous [6–12] and (ii) when the transition is continuous [13]. The first approach yields a 'ringing-effect' in the lower multipoles of the TT anisotropy power spectrum due to the abrupt matching of wave functions at the transition boundary [6–12], while the second method is devoid of any such effect due to a continuous transition between these two phases [13]. The lowering of the CMB quadrupole moment is evident in both these approaches to study a pre-inflationary radiation era. Alternatively, in ref. [46] the authors considered a scenario where the inflaton itself could have been in thermal equilibrium at some very early epoch possibly near the Planck era. The effect of this pre-inflationary dynamics was incorporated by considering a thermal rather than a vacuum state for the inflaton, i.e., by setting $\langle a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}'} \rangle = [\exp(k/T) - 1]^{-1} \delta^3(\mathbf{k} - \mathbf{k}')$, where $a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}}$ is the number operator for the inflaton modes and T is the inflaton comoving temperature. In contrast to the studies in refs. [6–13] this scenario led to an enhancement in power at low CMB multipoles corresponding to large angular scales. This suggests that there exist conflicting effects of a pre-inflationary radiation era: while the modified mode functions of the inflaton field lower the quadrupole moment, thermal initial conditions on the inflaton quanta tend to increase the power for the same.

In this article, we consider both these effects simultaneously unlike in earlier works that consider only one effect or the other. We find that the effects of the pre-inflationary era are only effective observationally if inflation lasts for the bare minimum number of e-folds required to solve the horizon and flatness problems. This is similar to the scenario when one considers a vacuum state with modified mode functions for the inflaton [6–13] (indicating that the effect of the thermal state is suppressed by the effect of the modified mode functions). Such 'just-enough' inflationary scenarios can be advocated from the fact that a large amount of inflation requires some fine-tunning [47, 48] and that string landscape models suffer from the $\eta-$problem [49] which does not allow them to sustain longer inflation.

The seminal work of Ford and Parker [50] showed that a pre-inflationary era, radiation or matter, can cure the infrared divergences which turn up in correlations of inflationary observables. We regard this as another motivation to study the consequences of a pre-inflationary radiation era in detail [7, 51, 52].

We begin in section 2 by discussing the assumptions and conditions we presume in our analysis. In section 3, we evaluate the primordial power spectrum of scalar perturbations by matching the inflaton mode functions in the inflationary era with those of the pre-inflationary radiation era, and by including a thermal distribution for the inflaton. In section 4, we use the WMAP and Planck data to determine the best-fit values or constraints on the duration of inflation and on the comoving temperature of the inflaton thermal distribution. We then conclude with a discussion of various issues in section 5.

2 Pre-inflationary radiation era

In this work we shall assume that before inflation began, the universe was described by a spatially flat FRW spacetime with small perturbations and was dominated by a radiation fluid whose equation of state was of the form $p = \rho/3$. At some epoch in the early universe, semiclassical general relativity and quantum field theory would have become valid and the calculations we shall present are applicable from this moment onwards. Below we discuss the assumptions that are relevant to the scenario that we are considering.

2.1 The little horizon problem

We presume an FRW metric associated with an isotropic and homogeneous universe prior to inflation in our analysis of cosmological perturbations during the pre-inflationary radiation dominated era. It requires a level of fine-tuning at the Planck epoch for this assumption of isotropy and homogeneity to be valid from $t_{\rm Pl}$ to t_i when inflation commences. The inflationary scenario too requires gradient energy to be sub-dominant on the horizon scale at the beginning of inflation.

If the Hubble parameter at the beginning of inflation is H_i and the scale factor at the beginning of inflation is a_i, then the physical size l_1 of the scale corresponding to H_i at the Planck time is

$$l_1 = \frac{H_i^{-1} a(t_{\rm Pl})}{a_i} \ . \tag{2.1}$$

If l_2 is the physical size of the horizon at the Planck time, then $l_2 = 1/H(t_{\rm Pl})$, and

$$\frac{l_1}{l_2} = \frac{a_{\rm Pl} H_{\rm Pl}}{a_i H_i} = \sqrt{\frac{M_{\rm Pl}}{H_i}} \ . \tag{2.2}$$

So, assuming $H_i \sim 10^{-4} M_{\rm Pl}$, where $M_{\rm Pl}$ is the reduced Planck mass, we need to assume that at the Planck epoch, the universe was isotropic and homogeneous on a length scale which is $\mathcal{O}(100)$ times larger than the Planck length for the FRW metric to be valid till t_i. If the energy scale of inflation or H_i is lower, one shall require even more fine-tuning at early times.

2.2 The little flatness problem

The dimensionless curvature density parameter is defined by $\Omega_K = -K/(a^2 H^2)$, where K could be -1, 0 or +1. In our analysis below of perturbations in the pre-inflationary radiation dominated universe we presume that Ω_K is negligible. An upper bound on the curvature is also required to ensure that the universe does not collapse before the onset of inflation (if $K = +1$). Since Ω_K increases in a decelerating universe, can we justify ignoring it? Let $t_{\rm Pl}$ and t_i be the Planck time and the epoch when inflation starts. Suppose the pre-inflationary radiation era lasts from $t_{\rm Pl}$ to t_i, what is the maximum value of Ω_K at $t_{\rm Pl}$ if we want Ω_K to be ignorable before inflation?

In the pre-inflationary era $\Omega_r + \Omega_\phi + \Omega_K = 1$, where $\Omega_{r,\phi}$ refer to the radiation and inflaton field component of the energy density, and let us assume that we can ignore Ω_K if it is $\mathcal{O}(0.01)$, i.e., at the onset of inflation, we expect that $\Omega_r \approx \Omega_\phi \approx \mathcal{O}(1) \gg \Omega_K$. Then, since $\Omega_K \sim a^{-2}$ for a radiation dominated universe

$$\Omega_K(t_{\rm Pl}) \approx \Omega_K(t_i) \left(\frac{H_i}{M_{\rm Pl}} \right) \ , \tag{2.3}$$

which, assuming $H_i \sim 10^{-4} M_{\rm Pl}$, turns out to be 10^{-6}. If the energy scale of inflation is lower, the fine-tuning problem gets more serious.

2.3 Local thermodynamic equilibrium

The stress tensor can be evaluated for any collection of particles but when the distribution function of the collection of particles is close to its form in thermodynamic equilibrium, the fluid approximation is valid and the stress tensor takes the simple form we use in the Friedmann equations in cosmology. In a pre-inflationary era, is the fluid approximation valid?

Was there enough time before inflation so that enough collisions could have happened that caused local thermodynamic equilibrium? That is, in the pre-inflationary era, how does the mean free path compare with the Hubble distance?

If a marginal coupling g with a vertex with three external lines contributes to 2-2 scattering process which sets the equilibrium, then, assuming that the coupling $g \approx \mathcal{O}(10^{-1/2})$ (the typical value of the Standard Model gauge couplings at the Planck scale), the ratio

$$\frac{t_{\text{coll}}}{t_{\text{Hubble}}} \sim \frac{T}{g^4 M_{\text{Pl}}} , \qquad (2.4)$$

is smaller than unity only when the "temperature" is hundred times smaller than the Planck mass which corresponds to $H/M_{\text{Pl}} \sim 10^{-4}$. If the Hubble parameter before inflation is much smaller than this, pre-inflationary relativistic particles do get enough time to attain thermal equilibrium and the fluid limit; if not, the fluid approximation is not valid.[2]

As an important aside, let us see what happens if the equilibrium is set by a gravitational interaction. The leading gravitational interaction is a dimension five operator coupling a graviton to, say, two scalars and is suppressed by the Planck mass

$$\mathcal{L}_{\text{int}} \supseteq c \frac{\mathcal{O}_5}{M_{\text{Pl}}} , \qquad (2.5)$$

which implies $\sigma(E) \sim c^4 E^2/M_{\text{Pl}}^4$ for 2-2 scattering mediated by a graviton. Hence

$$\frac{t_{\text{coll}}}{t_{\text{Hubble}}} \sim \frac{M_{\text{Pl}}^3}{T^3 c^4} , \qquad (2.6)$$

so that the mean collision time is smaller than the Hubble time only for super-Planckian temperatures. Thus, gravity mediated interactions can cause thermal equilibrium at sub-Planckian temperatures only if $c > 1$.

Further exploring the scenario where $t_{\text{coll}} > t_{\text{Hubble}}$ in the pre-inflationary radiation era, we could assume that somehow at the Planck time the universe was in local thermodynamic equilibrium but soon went out of equilibrium. Then, the distribution function of the relativistic particles is frozen in the equilibrium form while the relativistic particles form a hot decoupled relic radiation. Can the perturbations around the equilibrium distribution function be treated in the fluid approximation? Had the particles forming the pre-inflationary stuff been non-relativistic (like CDM), the fluid approximation would still have been valid for perturbations and we could have described them by just two variables: the density contrast δ and the peculiar velocity $\mathbf{v}$ (see, e.g., section 4.5 of ref. [55]). But they are relativistic, and so the fluid approximation does not hold good for the perturbations. In particular, the ideal fluid approximation breaks down for them as they cause anisotropic stresses (see eq. 5.33 of ref. [55]). Note that for photons and massless neutrinos at decoupling, the anisotropic stress is small because at the epoch of decoupling, the universe was already matter dominated.

Below we assume that the scale of inflation is low enough that the radiation has sufficient time to thermalise before inflation commences and so the fluid approximation is valid.

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[2]However if we lower the energy scale of inflation the fine-tuning required to have a pre-inflationary radiation dominated universe described by spatially flat FRW universe shall increase, unless the universe has a non-trivial topology [53]. (Quantum creation of a universe with non-trivial topology has been considered in ref. [54].)

2.4 Small Φ

To evaluate the scalar primordial power spectrum, we need to find the evolution of the comoving curvature perturbation $\mathcal{R}$, which during inflation takes the form

$$\mathcal{R} = \Phi + \frac{H}{\dot{\phi}_0} \delta\phi \,, \tag{2.7}$$

where Φ is the metric perturbation and the inflaton field $\phi(\mathbf{x}, t)$ is decomposed into a "classical" part $\phi_0(t)$ (which is its background value) and a quantum fluctuating part $\delta\phi(\mathbf{x}, t)$ as

$$\phi(\mathbf{x}, t) = \phi_0(t) + \delta\phi(\mathbf{x}, t). \tag{2.8}$$

For now we work in the conformal Newtonian gauge. During inflation, the metric perturbation Φ is negligible as compared to $\delta\phi$ (see figure (6.8) and section 6.5.2 of ref. [55]) and it becomes non-negligible only as inflation ends. Furthermore, for a radiation dominated universe, [56]

$$\Phi_k \sim \begin{cases} \text{constant,} & \text{for} \quad (\text{super} - \text{Hubble}), \\ \frac{\sin x}{x^2}, & \text{for} \quad (\text{sub} - \text{Hubble}) \,, \end{cases}$$

where $x = k\tau$ and τ is the conformal time. So we presume that Φ_k of the pre-inflationary radiation era dies down. This may not be strictly true for modes that enter the horizon just before inflation begins — for such modes we presume the super-Hubble value of Φ_k is small. Then the contribution of any pre-inflationary Φ_k can be ignored during inflation. Given the above, the scalar power spectrum during inflation is determined by the quantum fluctuations of $\delta\phi$ only and this is the quantity whose evolution we follow in the next section. In section 3, we shall work with a gauge invariant variable $\delta\varphi^{\text{gi}}$ which equals the field fluctuation in the conformal Newtonian gauge and shall perform matching of this variable at the transition from a radiation dominated universe to an inflaton dominated universe. We will find the power spectrum of $\mathcal{R}$ from the power spectrum of $\delta\phi^{\text{gi}}$. We shall also assume an instantaneous transition from a pre-inflationary radiation era to an inflationary era.

2.5 Initial conditions for the mode functions

When we have a pre-inflationary radiation dominated era, at early enough times, modes that are outside the horizon during the radiation era become subhorizon as time passes. The modes of cosmological interest enter the horizon before the inflationary era commences. We apply initial conditions corresponding to plane waves with a positive frequency for these modes, and also argue below that this is justified for modes that enter the horizon at the very end of the radiation dominated era.

2.6 Thermal initial state

If there is a pre-inflationary radiation dominated era, apart from the change in the inflaton mode functions, the state of the inflaton quanta could also be modified due to thermal effects as we now argue.

We may picture the energy density of the pre-inflationary universe as including contributions from (i) a species of relativistic particles (which form a fluid with an equation of state of the form $p = \rho/3$ with ρ falling as a^{-4}), and (ii) a coherent scalar field ϕ whose energy density does not dilute. Then, at some stage, the energy density of the radiation falls below the energy density of the scalar field and the universe begins inflating. One can

assume [46] that the quanta of the inflaton fluctuations $\delta\phi$ decouples from the rest of the plasma at some time t_d before inflation begins. After decoupling, the quanta of $\delta\phi$ travel along geodesics in the spacetime so that the distribution function $f(t, \mathbf{x}, \mathbf{p})$ is conserved (just like for collisionless dark matter; also, notice that $\mathbf{p}$ is the physical momentum). Assuming that the decoupling happens suddenly at a temperature $\mathcal{T}_d$, the frozen distribution function is the equilibrium distribution function f_{eq} at the epoch of decoupling: $f_{\text{d}} = f_{\text{eq}}(t_d, p_d)$, where, p_d is the physical momentum of the particle at the epoch of decoupling. Then, for the essentially non-interacting gas of (nearly) massless inflatons, the distribution function after decoupling is given by

$$f(t, \mathbf{x}, \mathbf{p}) = \frac{1}{\exp\left(\frac{a(t)p(t)}{a(t_d)\mathcal{T}(t_d)}\right) - 1} \, , \tag{2.9}$$

which has the same form as the equilibrium mean occupation number for a relativistic species with the temperature

$$\mathcal{T}(t) = \frac{\mathcal{T}(t_d)a(t_d)}{a(t)} \, , \tag{2.10}$$

even though the species $\delta\phi$ has fallen out of equilibrium. Notice that, just like for any decoupled species, this "temperature" falls strictly as a^{-1} unlike the temperature of a species which is in equilibrium (for which the relation between $\mathcal{T}$ and a depends on the number of relativistic degrees of freedom). Defining the comoving temperature T by $T = a(t)\mathcal{T}(t) = \mathcal{T}(t_d)a(t_d)$, the comoving temperature can be constrained [46], as explained below.

In this scenario, the modes of the quantum field $\delta\phi$ are not expected to be in a vacuum state but in a thermal state. This causes the scalar Primordial Power Spectrum to become [46]

$$P_{\mathcal{R}}(k) = A_s \left(\frac{k}{k_P}\right)^{n_s - 1} \coth\left(\frac{k}{2T}\right) \, , \tag{2.11}$$

and T being the comoving temperature introduces a length scale in the power spectrum. The observable k range is taken to be 10^0 Mpc^{-1} to 10^{-5} Mpc^{-1} and the presence of the coth factor in the above equation increases the power in CMB anisotropies at large angular scales. In order to not substantially affect the power spectrum over the observable k range, one requires the denominator in the coth function to be smaller than the present Hubble scale H_0, this constrains the comoving temperature [46]

$$T \leq 4.2 H_0 \approx 10^{-3} \text{Mpc}^{-1} \, . \tag{2.12}$$

Let the mode k_0 be such that $k_0 = a_0 H_0$ (i.e. it is crossing the Hubble radius today). If this mode exited the Hubble radius during inflation at an epoch t_*, then $T = \mathcal{T}_* a_*$, where, $\mathcal{T}_*$ and a_* denote the physical temperature and scale factor at $t = t_*$ when the Hubble parameter is H_*. Then, the above relation implies that $\mathcal{T}_* \leq 4.2 H_*$, using $a_* H_* = a_0 H_0$ (and setting $a_0 = 1$). At the onset of inflation $\rho_r = \rho_\phi$ where $\rho_\phi \approx V$ (slow-roll approximation) and $\rho_r = \frac{\pi^2 g_*}{30} \mathcal{T}_\gamma^4$ (g_* is the number of relativistic species in the pre-inflationary plasma and $\mathcal{T}_\gamma$ is the physical temperature of the pre-inflationary radiation at the epoch of onset of inflation). Assuming the physical temperature of the inflaton at the onset of inflation $\mathcal{T}_i = \mathcal{T}_\gamma$,[3] we get

$$\mathcal{T}_i = \left(\frac{30}{\pi^2 g_*}\right)^{1/4} V^{1/4} \, . \tag{2.13}$$

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[3]This assumption is valid when there is no entropy production between t_d and t_i.

The condition $\mathcal{T}_* \leq 4.2 H_*$ implies, using the fact that $H_* = \left(\frac{V^{1/4}}{M_{\mathrm{Pl}}} \right) V^{1/4}$,

$$\mathcal{T}_* \ll V^{1/4} \ . \tag{2.14}$$

Eqs. (2.13) and (2.14) together imply that in order to not affect the CMB temperature anisotropies at large angular scales, one needs more than the minimum amount of inflation (determined by when the mode k_0 left the Hubble radius during inflation) in such a scenario [46]. (Moreover, the amount of power at low ℓ values in the temperature anisotropy and B-mode polarization of CMB also increases if gravitons too were in thermal equilibrium at the Planck era [57–59].)

While the constraint according to ref. [46] on the comoving temperature of the inflaton quanta is $T \lesssim 10^{-3} \mathrm{Mpc}^{-1}$, the detailed analysis of section 4 in the present work has improved this constraint to $T \lesssim 10^{-4} \mathrm{Mpc}^{-1}$.

3 Evolution of perturbations

Having set up the basic scenario in section 2, one now has to solve for the metric perturbations which we present here in a gauge invariant form. We follow ref. [60] for our analysis. Considering only the scalar perturbations of the metric, the most general form of the perturbed spatially flat metric in conformal coordinates takes the form

$$g_{\mu\nu} \equiv g^0_{\mu\nu} + \delta g_{\mu\nu} = a^2(\tau) \begin{pmatrix} 1 + 2A & -\partial_i B \\ -\partial_i B & -(1 - 2\psi)\,\delta_{ij} - 2\partial_i\partial_j E \end{pmatrix}, \tag{3.1}$$

As these scalar perturbations are not gauge invariant quantities, it is useful to construct gauge invariant variables (known as Bardeen potentials) out of these metric perturbations as

$$\Phi = A + \mathcal{H}(B - E') + (B - E')', \tag{3.2}$$
$$\Psi = \psi - \mathcal{H}(B - E'), \tag{3.3}$$

where $\mathcal{H} = \frac{a'}{a} = aH$ and $X' \equiv \frac{\partial X}{\partial \tau}$. Absence of anisotropic stress in the stress-energy tensor puts a constraint on these gauge invariant quantities yielding $\Phi = \Psi$, irrespective of any particular choice of gauge. This is the quantity Φ introduced in section 2.4.

The fluctuation $\delta\phi(\mathbf{x}, t)$ in eq. (2.8) is not a gauge invariant quantity and a gauge invariant perturbation of the inflaton field $\delta\varphi^{\mathrm{gi}}(\mathbf{x}, t)$ can be constructed with the metric fluctuations as

$$\delta\varphi^{\mathrm{gi}} = \delta\phi + \phi_0'(B - E'). \tag{3.4}$$

In this perturbed background the equation of motion of the gauge invariant inflaton perturbation, taking into account $\Phi = \Psi$, is

$$\delta\varphi^{\mathrm{gi}\prime\prime} + 2aH\delta\varphi^{\mathrm{gi}\prime} - \nabla^2\delta\varphi^{\mathrm{gi}} + V_{,\phi\phi}\,a^2\delta\varphi^{\mathrm{gi}} = 4\phi_0'\Phi' - 2V_{,\phi}\,a^2\Phi\,. \tag{3.5}$$

In momentum space, we have

$$\delta\varphi_k^{\mathrm{gi}\prime\prime} + 2\mathcal{H}\delta\varphi_k^{\mathrm{gi}\prime} + k^2\delta\varphi_k^{\mathrm{gi}} + V_{,\phi\phi}\,a^2\delta\varphi_k^{\mathrm{gi}} = 4\phi_0'\Phi_k' - 2V_{,\phi}\,a^2\Phi_k. \tag{3.6}$$

Thermal effects can modify the equation of motion for the homogeneous and non-zero momentum modes by generating additional terms in the effective potential [61–64], even with the

frozen thermal distribution of inflaton quanta. One expects a mass correction, for example, $\sim \lambda T^2/a^2$ for a $\lambda\phi^4$ potential. (Recall that T/a is the physicaltemperature.) We compare this with k^2/a^2. For modes within the horizon at the beginning of inflation $k \geq a_i H_i$ and $k/a_i \geq H_i \sim \mathcal{T}_i^2/M_{\rm Pl} > \sqrt{\lambda}\mathcal{T}_i$ if λ is sufficiently small as we shall presume. Then for these modes of interest one can ignore these thermal correction to the potential in the equations of motion. Nevertheless one can treat the potential V above as including thermal corrections.

3.1 Subhorizon primordial perturbations in the pre-inflationary radiation era

The evolution of the scalar field perturbations in the pre-inflationary radiation era is governed by eq. (3.6). From the discussion in section 2.4 we ignore the r.h.s. of eq. (3.6). For a radiation dominated universe, $a = a_1 \left(\frac{t}{t_1}\right)^{1/2}$ and so the scale factor in terms of conformal time is

$$a = \left[\tau + \tau_1 - \frac{2t_1}{a_1}\right]\frac{a_1^2}{2t_1} . \tag{3.7}$$

If we now set τ_1 to $\frac{2t_1}{a_1}$ and choose t_1 to represent the epoch of transition from the pre-inflationary radiation era to inflation, i.e. t_i, then

$$a = \frac{a_i^2}{2t_i}\tau . \tag{3.8}$$

We restrict ourselves to a model of inflation in which the second potential slow-roll parameter η_V (defined to be $M_{\rm Pl}^2 V_{\phi\phi}/V$) is negative as, for example, for a potential of the form $V(\phi) = V_0 - m^2\phi^2/2$. (This allows us to recast the equations below in a more convenient form for solving. This assumption does not affect our results as mentioned in section 3.1.1.) One can then write eq. (3.6) as

$$\delta\varphi_k^{\rm gi\prime\prime} + 2\mathcal{H}\delta\varphi_k^{\rm gi\prime} + (k^2 - \tilde{c}^2\tau^2)\delta\varphi_k^{\rm gi} = 0, \tag{3.9}$$

where

$$\tilde{c}^2 = -\eta_V \frac{V}{M_{\rm Pl}^2}\frac{a_i^4}{4t_i^2} . \tag{3.10}$$

Below we shall take $\tilde{c}$ to be positive. Redefining the field as $\chi_k = a\delta\varphi_k^{\rm gi}$ one gets

$$\chi_k'' + (k^2 - \tilde{c}^2\tau^2)\chi_k = 0. \tag{3.11}$$

We rewrite the above equation as

$$\frac{d^2\chi_k}{dz^2} + \left(\nu + \frac{1}{2} - \frac{1}{4}z^2\right)\chi_k = 0, \tag{3.12}$$

where $\nu + \frac{1}{2} = -\frac{k^2}{2\tilde{c}}$ and $z = i\sqrt{2\tilde{c}}\,\tau$ and considering $\tilde{a} = -(\nu + \frac{1}{2}) = \frac{k^2}{2\tilde{c}}$ we get

$$\frac{d^2\chi_k}{dz^2} - \left(\frac{1}{4}z^2 + \tilde{a}\right)\chi_k = 0 , \tag{3.13}$$

(compare with, e.g., eq. (19.1.2) of ref. [65]). The even and odd solutions of the above equation are given in eqs. (19.2.5) and (19.2.6) of ref. [65] which are as follows

$$\chi_k^1 = 1 + \tilde{a}\frac{z^2}{2!} + \left(\tilde{a}^2 + \frac{1}{2}\right)\frac{z^4}{4!} + \left(\tilde{a}^3 + \frac{7}{2}\tilde{a}\right)\frac{z^6}{6!} + \left(\tilde{a}^4 + 11\tilde{a}^2 + \frac{15}{4}\right)\frac{z^8}{8!}$$
$$+ \left(\tilde{a}^5 + 25\tilde{a}^3 + \frac{211}{4}\tilde{a}\right)\frac{z^{10}}{10!} + \cdots$$
$$\chi_k^2 = z + \tilde{a}\frac{z^3}{3!} + \left(\tilde{a}^2 + \frac{3}{2}\right)\frac{z^5}{5!} + \left(\tilde{a}^3 + \frac{13}{2}\tilde{a}\right)\frac{z^7}{7!} + \left(\tilde{a}^4 + 17\tilde{a}^2 + \frac{63}{4}\right)\frac{z^9}{9!}$$
$$+ \left(\tilde{a}^5 + 35\tilde{a}^3 + \frac{531}{4}\tilde{a}\right)\frac{z^{11}}{11!} + \cdots \tag{3.14}$$

Hence the asymptotic forms of the above two solutions with $\tilde{a}$ large are

$$\chi_k^1 \approx 1 + \tilde{a}\frac{z^2}{2!} + \tilde{a}^2\frac{z^4}{4!} + \tilde{a}^3\frac{z^6}{6!} + \tilde{a}^4\frac{z^8}{8!} + \tilde{a}^5\frac{z^{10}}{10!} + \cdots = \cosh(\sqrt{\tilde{a}}z)$$
$$\chi_k^2 \approx z + \tilde{a}\frac{z^3}{3!} + \tilde{a}^2\frac{z^5}{5!} + \tilde{a}^3\frac{z^7}{7!} + \tilde{a}^4\frac{z^9}{9!} + \tilde{a}^5\frac{z^{11}}{11!} + \cdots = \frac{1}{\sqrt{\tilde{a}}}\sinh(\sqrt{\tilde{a}}z) \tag{3.15}$$

If we now define v_1 and v_2 by

$$v_1 = \chi_k^1 + \sqrt{\tilde{a}}\chi_k^2 \approx e^{\sqrt{\tilde{a}}z} = e^{ik\tau}$$
$$v_2 = \chi_k^1 - \sqrt{\tilde{a}}\chi_k^2 \approx e^{-\sqrt{\tilde{a}}z} = e^{-ik\tau}, \tag{3.16}$$

the expression for $\chi(k)$ during the radiation era is

$$\chi_k(\tau) = c_1(k)v_1 + c_2(k)v_2, \tag{3.17}$$

which in the sub-horizon limit becomes

$$\chi_k(\tau) \approx c_1(k)e^{ik\tau} + c_2(k)e^{-ik\tau} . \tag{3.18}$$

To obtain the Minkowski spacetime solutions in the sub-horizon limit, we choose $c_2(k) = \frac{1}{\sqrt{2k}}$ and $c_1(k) = 0$ (for all k). Then, for sub-horizon modes

$$\delta\varphi_{\text{rad}}^{\text{gi}}(k,\tau) \approx \frac{1}{a(\tau)\sqrt{2k}}e^{-ik\tau} \tag{3.19}$$

3.1.1 Justification for large $\tilde{a}$

To obtain the asymptotic expressions in eq. (3.15) we assumed that $\tilde{a}$ is large. Since $\tilde{a} = \frac{k^2}{2\tilde{c}}$, this is equivalent to assuming that $k^4 \gg 4\tilde{c}^2$, or

$$k^4 \gg 4|\eta_V|\frac{V}{M_{\text{Pl}}^4}\frac{a_i^4}{4t_i^2}M_{\text{Pl}}^2 . \tag{3.20}$$

Since $t = 1/(2H)$ for a radiation dominated universe, the above gives

$$k^4 \gg 4|\eta_V|\frac{V}{M_{\text{Pl}}^4}a_i^4 H_i^2 M_{\text{Pl}}^2 . \tag{3.21}$$

We need to consider modes for which $k \gtrsim a_i H_i$, i.e. modes that enter the horizon during the pre-inflationary radiation era. For the smallest $k = a_i H_i$ the above condition then gives

$$H_i^2 \gg 4|\eta_V| \frac{V}{M_{\rm Pl}^2} \ . \tag{3.22}$$

At the epoch of transition $H_i^2 = \frac{V}{3M_{\rm Pl}^2}$, which implies that the condition for having large $\tilde{a}$ is that

$$|\eta_V| \ll \frac{1}{12} \ . \tag{3.23}$$

Thus, if the potential is sufficiently flat, considering $\tilde{a}$ to be large can be justified for $k \gtrsim a_i H_i$. This analysis justifies the plane wave form of the mode functions at t_i even for modes which were just entering the horizon at the onset of inflation.

Arguments similar to the above can be used to show that for a potential with η_V of either sign, $k^2 \gg \tilde{c}\tau^2$ in eq. (3.9) at τ_i for modes of interest if $|\eta_V| \ll \frac{1}{3}$, thereby justifying the plane wave form of the mode functions in the pre-inflationary era at t_i.

3.2 Evolution of primordial perturbations during inflation

Using the background equation for the slow-rolling inflaton field, i.e. $V_{,\phi}\, a^2 \approx -2\mathcal{H}\phi_0'$, eq. (3.6) can be written as

$$\delta\varphi_k^{\rm gi\prime\prime} + 2\mathcal{H}\delta\varphi_k^{\rm gi\prime} + k^2\delta\varphi_k^{\rm gi} + V_{,\phi\phi}\, a^2 \delta\varphi_k^{\rm gi} = 4\phi_0'\Phi_k' + 4\mathcal{H}\phi_0'\Phi_k \ . \tag{3.24}$$

The conformal time in quasi-de Sitter space is $\tau = -\frac{1}{(1-\epsilon)\mathcal{H}}$, where the Hubble slow-roll parameter $\epsilon \equiv -\frac{\dot{H}}{H^2} = 4\pi G \frac{\phi_0'^2}{\mathcal{H}^2}$. Now the $0-i^{\rm th}$ component of the Einstein equation can be written during inflation as

$$\Phi' + \mathcal{H}\Phi = 4\pi G\phi_0'\delta\varphi^{\rm gi}. \tag{3.25}$$

Using this in the R.H.S. of eq. (3.24) yields

$$\delta\varphi_k^{\rm gi\prime\prime} + 2\mathcal{H}\delta\varphi_k^{\rm gi\prime} + k^2\delta\varphi_k^{\rm gi} + V_{,\phi\phi}\, a^2 \delta\varphi_k^{\rm gi} = 16\pi G\phi_0'^2\delta\varphi_k^{\rm gi}. \tag{3.26}$$

Now, we can relate the last two terms in the above equation to the slow-roll parameters. The Hubble slow-roll parameter $\eta \equiv -\frac{\ddot{\phi_0}}{H\dot{\phi_0}}$ and the standard slow-roll parameters are $\epsilon_V \equiv \frac{M_{\rm Pl}^2}{2}\left(\frac{V'}{V}\right)^2$ and $\eta_V = M_{\rm Pl}^2\left(\frac{V''}{V}\right) = \frac{1}{3}\frac{a^2 V_{,\phi\phi}}{\mathcal{H}^2}$. In the slow-roll regime we have $\epsilon_V \approx \epsilon$ and $\eta_V \approx \eta + \epsilon$. Hence we can write the above equation as

$$\delta\varphi_k^{\rm gi\prime\prime} + 2\mathcal{H}\delta\varphi_k^{\rm gi\prime} + k^2\delta\varphi_k^{\rm gi} + (3\eta - \epsilon)\mathcal{H}^2\delta\varphi_k^{\rm gi} = 0. \tag{3.27}$$

It is convenient to redefine the field as $\chi_k = a\delta\varphi_k^{\rm gi}$, as we did before, whose Wronskian yields $\chi_k\chi_k^{*\prime} - \chi_k^*\chi_k' = i$ or $\chi_k\dot{\chi}_k^* - \chi_k^*\dot{\chi}_k = \frac{i}{a(t)}$. The equation of motion for χ_k during inflation is

$$\chi_k'' + \left[k^2 - \frac{a''}{a} + (3\eta - \epsilon)\mathcal{H}^2\right]\chi_k = 0. \tag{3.28}$$

In a quasi-de Sitter space $\frac{a''}{a} \approx \frac{1}{\tau^2}(2 + 3\epsilon)$ and $\mathcal{H} = -\frac{1}{\tau(1-\epsilon)}$. Thus in a quasi-de Sitter space the above equation can be written as

$$\chi_k'' + \left[k^2 - \frac{1}{\tau^2}(2 - 3\eta + 4\epsilon)\right]\chi_k = 0, \tag{3.29}$$

or as

$$\chi_k'' + \left[k^2 - \frac{1}{\tau^2} \left(\nu_\chi^2 - \frac{1}{4} \right) \right] \chi_k = 0, \tag{3.30}$$

where $\nu_\chi^2 \equiv \frac{9}{4} - 3\eta + 4\epsilon$. The solution of the above equation can be written as

$$\chi_k = \sqrt{-\tau} \left[\tilde{c}_1(k) H_{\nu_\chi}^{(1)}(-k\tau) + \tilde{c}_2(k) H_{\nu_\chi}^{(2)}(-k\tau) \right], \tag{3.31}$$

where $H_{\nu_\chi}^{(1)}$ and $H_{\nu_\chi}^{(2)}$ are the Hankel functions of the first and second kind.

Thus during the inflationary era the mode functions of the gauge invariant inflaton fluctuations have a solution

$$\delta\varphi_{\inf}^{\mathrm{gi}}(k,\tau) \equiv a(\tau)^{-1}\chi_k = a(\tau)^{-1}\sqrt{-\tau} \left[\tilde{c}_1(k) H_{\nu_\chi}^{(1)}(-k\tau) + \tilde{c}_2(k) H_{\nu_\chi}^{(2)}(-k\tau) \right]. \tag{3.32}$$

(Had we ignored the R.H.S. of eq. (3.24) due to small Φ_k, we would have obtained $\nu_\chi^2 \equiv \frac{9}{4} - 3\eta + 3\epsilon$.)

3.3 Mode function matching and the power spectrum

For a transition from a radiation dominated era to an inflationary era at $t = t_i$ the form of the scale factor changes as

$$a(t) = a_i(t/t_i)^{1/2}, \qquad t \le t_i \tag{3.33}$$

$$a(t) = a_i e^{H_i(t-t_i) + \frac{\dot{H}_i}{2}(t-t_i)^2}, \qquad t > t_i. \tag{3.34}$$

H_i is the scale factor at the time of the transition. $H_i = \frac{1}{2t_i}$. a and $\dot{a}$ are continuous at t_i. We have seen in the previous two sub-sections that the evolution of the gauge invariant inflaton fluctuations during the pre-inflationary radiation era and in the inflationary era is given by eq. (3.19) and eq. (3.32). We define $z_{R,I} \equiv -k\tau = \frac{k}{(1-\epsilon_{R,I})aH}$ where $\epsilon_{R,I}$ are the slow-roll parameter during the radiation and inflationary eras. $\epsilon_R = 2$ while $\epsilon_I \ll 1$. τ in the definition of z_R above is consistent with eq. (3.8). $z(k)$ is not continuous at the transition (unlike in some earlier works such as refs. [8–12] thereby giving somewhat different final expressions). Then

$$\delta\varphi_{\mathrm{rad}}^{\mathrm{gi}}(k,t) = a_{\mathrm{rad}}(t)^{-1}\chi_{\mathrm{rad}}(z_R), \qquad t \le t_i \tag{3.35}$$

$$\delta\varphi_{\inf}^{\mathrm{gi}}(k,t) = a_{\inf}(t)^{-1} \left[C_1(k)u_{\inf}(z_I) + C_2(k)u_{\inf}^*(z_I) \right], \qquad t > t_i, \tag{3.36}$$

where

$$\chi_{\mathrm{rad}}(z_R) = \frac{1}{\sqrt{2k}}e^{-ik\tau} = \frac{1}{\sqrt{2k}}e^{iz_R}, \tag{3.37}$$

$$u_{\inf}(z_I) = \sqrt{\frac{\pi z_I}{4k}} H_{\nu_\chi}^{(1)}(z_I), \tag{3.38}$$

$$u_{\inf}^*(z_I) = \sqrt{\frac{\pi z_I}{4k}} H_{\nu_\chi}^{(2)}(z_I), \tag{3.39}$$

and $C_1 = \sqrt{\frac{4}{\pi}}\tilde{c}_1$ and $C_2 = \sqrt{\frac{4}{\pi}}\tilde{c}_2$. A subscript k for $\chi_{\mathrm{rad}}(z_R)$ and $u_{\inf}(z_I)$ is implicit. The Wronskian of $u_{\inf}(z_I)$ gives $|C_1|^2 - |C_2|^2 = 1$. The task is to determine the coefficients $C_1(k)$ and $C_2(k)$.

We demand that the wavefunction of gauge invariant inflaton fluctuation and its time derivative remain continuous at the time of the transition, i.e.

$$\delta\varphi_{\text{rad}}^{\text{gi}}(t_i) = \delta\varphi_{\text{inf}}^{\text{gi}}(t_i),$$
$$\delta\dot\varphi_{\text{rad}}^{\text{gi}}(t_i) = \delta\dot\varphi_{\text{inf}}^{\text{gi}}(t_i). \tag{3.40}$$

Following eq. (3.35) and eq. (3.36) we get

$$\delta\dot\varphi_{\text{rad}}^{\text{gi}}(t_i) = \frac{\dot\chi_{\text{rad}}(t_i)}{a(t_i)} - \frac{\chi_{\text{rad}}(t_i)}{2t_i a(t_i)}, \tag{3.41}$$

$$\delta\dot\varphi_{\text{inf}}^{\text{gi}}(t_i) = \frac{C_1(k)\dot u_{\text{inf}}(t_i) + C_2(k)\dot u_{\text{inf}}^*(t_i)}{a(t_i)} - \frac{C_1(k)u_{\text{inf}}(t_i) + C_2(k)u_{\text{inf}}^*(t_i)}{2t_i a(t_i)}. \tag{3.42}$$

Using the matching conditions given in eq. (3.40) one gets

$$\chi_{\text{rad}}|_{t=t_i} = C_1(k)u_{\text{inf}}|_{t=t_i} + C_2(k)u_{\text{inf}}^*|_{t=t_i}, \tag{3.43}$$

$$\dot\chi_{\text{rad}}|_{t=t_i} = C_1(k)\dot u_{\text{inf}}|_{t=t_i} + C_2(k)\dot u_{\text{inf}}^*|_{t=t_i}, \tag{3.44}$$

where we have used eq. (3.43) to simplify expressions and get eq. (3.44). Alternatively we could have matched $a\delta\varphi^{\text{gi}}$ and its derivative at the transition. Solving these two equations simultaneously and using the Wronskian $u_{\text{inf}}\dot u_{\text{inf}}^* - u_{\text{inf}}^*\dot u_{\text{inf}} = \frac{i}{a(t)}$ (yielding $|C_1|^2 - |C_2|^2 = 1$) one gets

$$C_1(k) = ia(t_i)\left(u_{\text{inf}}^*\dot\chi_{\text{rad}} - \dot u_{\text{inf}}^*\chi_{\text{rad}}\right)|_{t=t_i}, \tag{3.45}$$

$$C_2(k) = ia(t_i)\left(\dot u_{\text{inf}}\chi_{\text{rad}} - u_{\text{inf}}\dot\chi_{\text{rad}}\right)|_{t=t_i}. \tag{3.46}$$

Now to determine $C_1(k)$ and $C_2(k)$ we need $\dot\chi_{\text{rad}}$, $\dot u_{\text{inf}}$ and $\dot u_{\text{inf}}^*$ at t_i. To obtain expressions for these three quantities we notice that $\dot z = -\frac{k}{a}$ and the derivatives of the Hankel functions are (eq. (5.3.5) and eq. (5.4.9) of ref. [66]):

$$\frac{d}{dz}H_\nu^{(1,2)}(z) = \frac{\nu H_\nu^{(1,2)}(z)}{z} - H_{\nu+1}^{(1,2)}(z), \tag{3.47}$$

where ν is an arbitrary order. Using the above equations one gets

$$\dot\chi_{\text{rad}}(z_R) = -\frac{i}{a}\sqrt{\frac{k}{2}}e^{iz_R}, \tag{3.48}$$

$$\dot u_{\text{inf}}(z_I) = \frac{1}{a}\sqrt{\frac{\pi k}{4z_I}}\left[z_I H_{\nu_\chi+1}^{(1)}(z_I) - \left(\nu_\chi + \frac{1}{2}\right)H_{\nu_\chi}^{(1)}(z_I)\right], \tag{3.49}$$

$$\dot u_{\text{inf}}^*(z_I) = \frac{1}{a}\sqrt{\frac{\pi k}{4z_I}}\left[z_I H_{\nu_\chi+1}^{(2)}(z_I) - \left(\nu_\chi + \frac{1}{2}\right)H_{\nu_\chi}^{(2)}(z_I)\right]. \tag{3.50}$$

Thus the co-efficients are

$$C_1(k) = ie^{iz_R}\sqrt{\frac{\pi}{8z_I}}\left[\left(\nu_\chi + \frac{1}{2} - iz_I\right)H_{\nu_\chi}^{(2)}(z_I) - z_I H_{\nu_\chi+1}^{(2)}(z_I)\right]\Big|_{t_i}, \tag{3.51}$$

$$C_2(k) = -ie^{iz_R}\sqrt{\frac{\pi}{8z_I}}\left[\left(\nu_\chi + \frac{1}{2} - iz_I\right)H_{\nu_\chi}^{(1)}(z_I) - z_I H_{\nu_\chi+1}^{(1)}(z_I)\right]\Big|_{t_i}. \tag{3.52}$$

To check that $|C_1|^2 - |C_2|^2 = 1$ we obtain

$$|C_1(k)|^2 - |C_2(k)|^2 = -\frac{i\pi z_I}{4}\left[H^{(1)}_{\nu_\chi}(z_I) H^{(2)}_{\nu_\chi+1}(z_I) - H^{(2)}_{\nu_\chi}(z_I) H^{(1)}_{\nu_\chi+1}(z_I) \right]\Big|_{t_i} \quad (3.53)$$

Now using the relation in eq. (9.1.17) of ref. [67]),

$$H^{(1)}_{\nu+1}(z) H^{(2)}_\nu(z) - H^{(2)}_{\nu+1}(z) H^{(1)}_\nu(z) = -\frac{4i}{\pi z} \quad (3.54)$$

in the above equation we get $|C_1|^2 - |C_2|^2 = 1$.

As was discussed in section 2.4, during inflation, the curvature perturbation $\mathcal{R}$ receives most of its contribution from $\delta\phi$ and so we can use the above to find the primordial power spectrum. When inflation is preceded by a radiation era the power spectrum for the inflaton field fluctuations in the vacuum state is (using arguments similar to those in ref. [6])

$$\mathcal{P}_{\delta\phi}(k) = \mathcal{P}^{\mathrm{BD}}_{\delta\phi}|C_1 - C_2|^2 \,, \quad (3.55)$$

where $\mathcal{P}^{\mathrm{BD}}_{\delta\phi}$ corresponds to the standard power spectrum in vacuum presuming Bunch-Davies initial conditions on $\delta\varphi^{\mathrm{gi}}_{\mathrm{inf}}$. Then

$$\mathcal{P}_{\mathcal{R}}(k) = \mathcal{P}^{\mathrm{BD}}_{\mathcal{R}}|C_1 - C_2|^2 = A\left(\frac{k}{k_P}\right)^{n_s-1}|C_1 - C_2|^2, \quad (3.56)$$

where A and n_s are the amplitude and scalar spectral index respectively of the spectrum in a generic inflationary scenario, and k_P is the pivot scale. Thus the modification due to mode function-matching is a multiplicative factor of $|C_1 - C_2|^2$ which we will calculate now. We have

$$C_1(k) - C_2(k) = ie^{iz_R}\sqrt{\frac{\pi}{2z_I}}\left[\left(\nu_\chi + \frac{1}{2} - iz_I\right) J_{\nu_\chi}(z_I) - z_I J_{\nu_\chi+1}(z_I)\right]\Big|_{t_i}, \quad (3.57)$$

$$C_1^*(k) - C_2^*(k) = -ie^{-iz_R}\sqrt{\frac{\pi}{2z_I}}\left[\left(\nu_\chi + \frac{1}{2} + iz_I\right) J_{\nu_\chi}(z_I) - z_I J_{\nu_\chi+1}(z_I)\right]\Big|_{t_i}, \quad (3.58)$$

which yields

$$|C_1 - C_2|^2 = \frac{\pi}{2z_I}\left[z_I^2\left(J^2_{\nu_\chi}(z_I) + J^2_{\nu_\chi+1}(z_I)\right) - 2z_I\left(\nu_\chi + \frac{1}{2}\right) J_{\nu_\chi}(z_I) J_{\nu_\chi+1}(z_I) \right.$$
$$\left. + \left(\nu_\chi + \frac{1}{2}\right)^2 J^2_{\nu_\chi}(z_I)\right]\Bigg|_{t_i}. \quad (3.59)$$

We need to evaluate $z_I(k, t_i)$ to determine the above factor. We know that

$$z_I(k, t_i) = \frac{k}{(1 - \epsilon_I)a_i H_i}. \quad (3.60)$$

For the mode corresponding to the horizon size at the onset of inflation we have $k_i = a_i H_i$. For the largest mode of cosmological interest with wavenumber $k_0 = a_0 H_0 = H_0$ (for $a_0 = 1$), which leaves $N(k_0)$ e-foldings before inflation ends, we have

$$N(k_0) - N(k_i) = \ln(k_i/k_0). \quad (3.61)$$

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This gives

$$k_i = k_0 e^{-\delta N}, \tag{3.62}$$

where we have defined $\delta N \equiv N(k_i) - N(k_0)$. Then

$$z_I(k, t_i) = \frac{k}{(1 - \epsilon_I) H_0} e^{\delta N}. \tag{3.63}$$

Now if we also consider a thermal distribution of the scalar field quanta as in section 2.6 then the power spectrum will have an additional multiplicative factor and will be given by

$$\mathcal{P}_{\mathcal{R}}(k) = A'_s \left(\frac{k}{k_P} \right)^{n_s - 1} |C_1 - C_2|^2 \coth\left(\frac{k}{2T} \right), \tag{3.64}$$

where T is the comoving temperature of the scalar field quanta. We can normalize the above equation as

$$\mathcal{P}_{\mathcal{R}}(k) = A_s \left(\frac{k}{k_P} \right)^{n_s - 1} |C_1 - C_2|^2 \coth\left(\frac{k}{2T} \right) \left[|C_1(k_P) - C_2(k_P)|^2 \coth\left(\frac{k_P}{2T} \right) \right]^{-1} \tag{3.65}$$

so that $P_{\mathcal{R}}(k_P) = A_s$. In figure (1) we plot the form of the Bunch-Davies power spectrum and the modified power spectra of eqs. (3.56) and (3.65).

3.4 Infrared divergences

Before proceeding to the estimation of parameters for this model with a pre-inflationary radiation era, let us see whether the modified power spectrum is free of infrared divergences or not. The two-point correlation function G at coincident points is proportional to the power spectrum as

$$G \propto \int \frac{dk}{k} \mathcal{P}_{\mathcal{R}}. \tag{3.66}$$

For a generic inflationary scenario with no pre-inflationary era one has

$$\mathcal{P}_{\mathcal{R}} = A_s (k/k_P)^{n_s - 1}, \tag{3.67}$$

with $n_s - 1 = 2\eta_V - 6\epsilon_V$. This spectrum diverges when $k \to 0$. But in a scenario where the inflation is preceded by a radiation era, the mode-matching at the boundary brings up the multiplicative factor given in eq. (3.59). In the small k limit this factor will go as $\propto k^{2\nu_\chi - 1}$. Now we have $\nu_\chi \approx 3/2$. Thus, in such a case the correlation function will be

$$G \propto \int dk \, k^{1 + 2\eta_V - 6\epsilon_V} \tag{3.68}$$

which converges at $k \to 0$ (Note that the dependence of k in such a case matches with [7]). Also, if we consider the thermal distribution of the inflaton then it will bring up an extra factor of $\coth(2kT)$ which as $k \to 0$ will go as k^{-1}. Thus in such a case the correlation function will be

$$G \propto \int dk \, k^{2\eta_V - 6\epsilon_V}, \tag{3.69}$$

which is also IR convergent. Hence the power spectrum is infrared divergence free once we take into account a radiation era preceding inflation, a fact well known in the literature (for the standard power spectrum without the coth factor) [50].
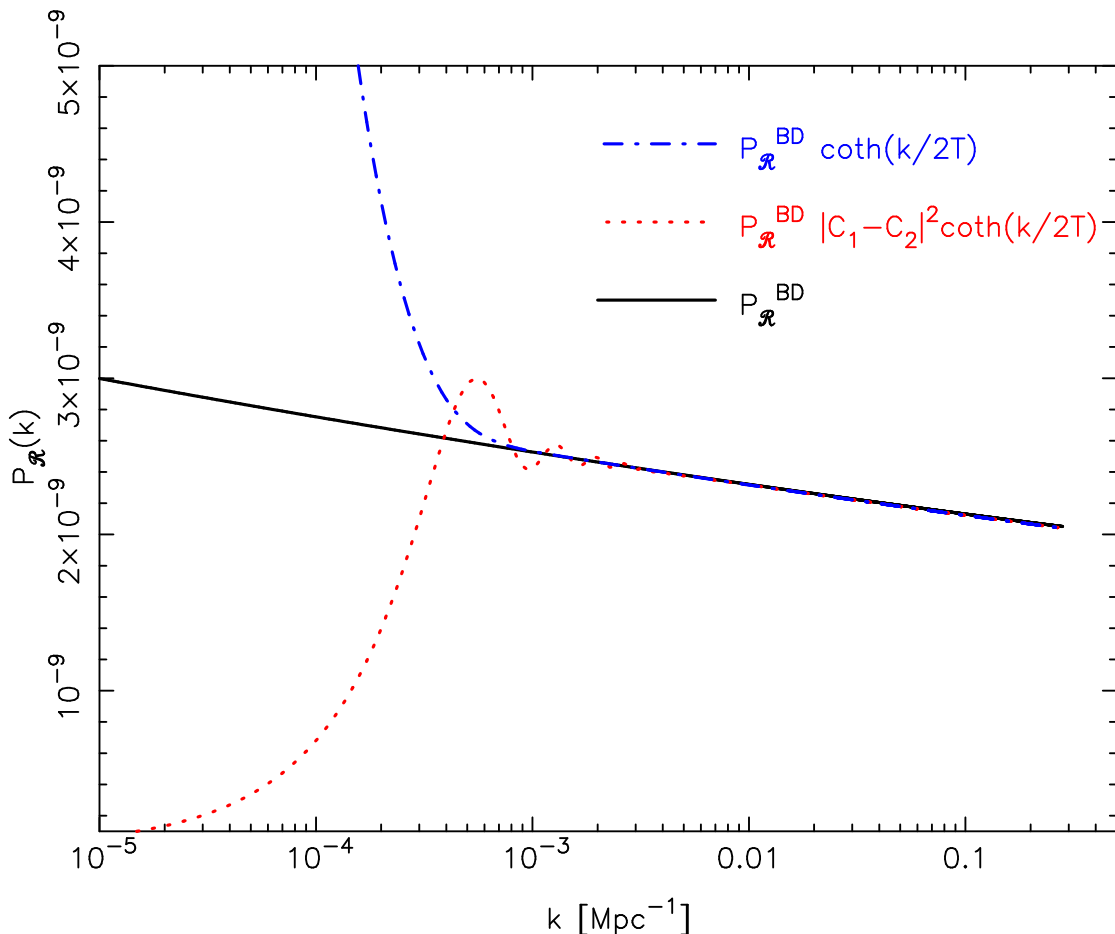
Figure 1. The effects of various factors in eq. (3.65) on the shape of the scalar Primordial Power Spectrum. While the coth term enhances the power at low k values, the factor of $|C_1 - C_2|^2$ due to non-trivial mode dynamics lowers it. We have used $\delta N = 0.081$ and $T = 1.286 \times 10^{-4}$ Mpc^{-1} (from table 2). The Bunch-Davies power law Primordial Power Spectrum has also been shown for reference.

4 Parameter estimation

In the inflationary scenario of our interest, the Primordial Power Spectrum (PPS) has two extra parameters δN, the number of e-foldings of inflation in excess of the standard minimum (which is, for example, 60 e-foldings for GUT scale inflation) and the comoving inflaton temperature T. We try to constrain the extra parameters of PPS from the WMAP nine year and Planck data in this section using Markov Chain Monte Carlo (MCMC) analysis and for this purpose use the publicly available code `COSMOMC` [68, 69].

`COSMOMC` uses a publicly available code `CAMB` [70] based a line of sight integration approach given in ref. [71] for computing the power spectra of CMB anisotropies for a set of cosmological parameters. Apart from the two extra parameter δN and T, characterizing the primordial power spectrum, we vary six standard parameters, namely, the physical densities of baryons ($\Omega_b h^2$) and dark matter ($\Omega_c h^2$), the dark energy density (Ω_Λ) or θ as defined later, the amplitude (A_S) and spectral index (n_s) of the primordial power spectrum (at pivot scale $k_P = 0.05$ Mpc^{-1}) and the optical depth of reionization (τ) in our MCMC analysis.

We modify `CAMB` such that it can incorporate the two extra parameters for PPS which

we have. In order to compute the likelihood from the power spectra of the CMB anisotropies for WMAP nine year and Planck data we use the likelihood codes provided by WMAP [72] and Planck team [73] respectively. For completeness a short description of the WMAP nine year and Planck likelihood codes and data is given below.

The WMAP nine year likelihood code and the methodology of parameter estimation is discussed in detail in ref. [74] and is not very different than what was outlined in ref. [75]. WMAP likelihood and nine year temperature and polarization data can be downloaded from ref. [72]. The code computes likelihoods for the TT, EE, TE and BB angular power spectra differently at low and high-l. The low-l ($l \leq 32$) TT likelihood is computed from the power spectrum estimated by Gibbs sampling and high-l ($l > 32$) TT likelihood is computed using an optimal quadratic estimator. The WMAP likelihood code computes the low-l ($l < 23$) polarization likelihood directly in the pixel space.

At present Planck has made only the temperature data publicly available which can be used alone or with a combination of other CMB data sets to constrain theoretical models. The likelihood code for Planck data can be downloaded from ref. [73] and a detailed description of it can be found in ref. [76]. Since Planck has more frequency channels spread over a wider range, higher angular resolution and better sensitivity as compared to WMAP it is far better equipped to deal with systematics like foregrounds. Higher angular resolution and better sensitivity of Planck allows us to use the angular power spectrum (temperature) up to $l = 2500$, and a higher number of frequency channels makes it possible to model foregrounds more accurately.

The Planck likelihood software (`Clik`) has a few different likelihoods modules, some of which are `CAMspec` for computing the TT likelihood at high-l (up to $l = 2500$), `commander` for computing the TT likelihood at low-l (from $l = 0$ to $l = 49$) and `lowlike` for computing the low-l ($l = 0$ to $l = 32$) polarization likelihood from the TT, EE, BB and TE power spectra (for polarization it uses WMAP nine year data).

The `CAMspec` module of the Planck likelihood code has 14 extra (nuisance) parameters which are used for modeling systematics like foregrounds, asymmetric beams, etc. In principle, these extra parameters also can be estimated from the same data set from which the cosmological parameters are estimated. However, we do not do that and instead fix the values of the `CAMspec` nuisance parameters to their values reported in refs. [76, 77], and vary only the standard six standard parameters and the extra parameters of PPS.

For our analysis we take the sum of the physical masses of the light neutrinos ($\sum_\nu$) as $0.6\,\mathrm{eV}$, the effective number of neutrinos (N_{eff}) as 3.046, the Helium mass fraction (Y_{He}) as 0.24 and the width of reionization as 0.5. For the case of spatially flat background cosmological models, as we consider here, either the Hubble parameter at present (H_0) or the dark energy density is considered as a fitting parameter and the other is computed from the flatness condition. In practice `COSMOMC` does not use H_0 or Ω_Λ as one of the six base parameters and rather uses θ, the ratio of the size of the sound horizon at decoupling (r_{dec}) and the angular diameter distance at decoupling (D_A), as one of the parameters.

The prior ranges which we use for parameter estimation are given in table 1. We found the prior ranges for the two extra parameters δN and T (in Mpc^{-1}) by considering a few test cases. For the rest of the cosmological parameters we use the same prior ranges as used in the literature [77].

As is clear from eq. (3.65), the primordial power spectrum depends on ϵ_I and ν_χ. We consider two values of ϵ_I, namely, 0.001 and 0. The choice of $\epsilon_I = 0.001$ with the presumed range of A_s in table 1 effectively sets $H_i/M_{Pl} \sim 10^{-6}$, or $V_i^{1/4} \sim 10^{15}\,\mathrm{GeV}$, while $\epsilon_I = 0$

S. No	Parameter	Prior	Description
1	$\Omega_b h^2$	$[0.005, 0.1]$	Baryon density today
2	$\Omega_c h^2$	$[0.001, 0.99]$	Cold dark matter density today
3	100θ	$[0.55, 10.0]$	$\approx 100 \times r_{dec}/D_A$ (CosmoMC)
4	$\ln[10^{10} A_s]$	$[2.7, 4.0]$	Primordial curvature perturbations at $k_0 = 0.05\,\mathrm{Mpc}^{-1}$
5	n_s	$[0.9, 1.1]$	Scalar spectrum power-law index at $k_0 = 0.05\,\mathrm{Mpc}^{-1}$
6	τ	$[0.01, 0.8]$	Thomson scattering optical depth due to reionization
7	δN	$[-2.000, 2.0]$	Number of extra e-foldings
8	T	$[0.0000001, 0.0004]$	Temperature in Mpc^{-1}

**Table 1**. Cosmological parameters and the prior ranges which we use in `COSMOMC`

corresponds to a very low energy scale of inflation, such as at the electroweak scale. These choices are consistent with the upper bound in section 2.3 on H_i from the requirement of thermalization before inflation so as to avail of the fluid approximation. ν_χ as defined in section 3.2 depends on ϵ and η during inflation. Since the power law factor in eq. (3.65) is determined by the slow-roll dynamics of the inflaton, we still have $n_s = 1 - 4\epsilon + 2\eta$ during inflation. For the above values of ϵ_I we vary n_s and can obtain the corresponding values of η.

For the case of $\epsilon_I = 0.001$, the extreme values of the prior range of n_s according to table 1, i.e. 0.9 and 1.1, lead to η_V lying between -0.047 and 0.053 which includes values which are not too small as compared to the upper bound of $1/12 = 0.083$ in eq. (3.23). But the best fit value and 1σ range of n_s (in table 2) correspond to values of η_V that satisfy the upper bound. We have also checked that restricting the prior range of n_s to $[0.954, 1.034]$, which corresponds to $|\eta_V| < 0.02$ (for $\epsilon_I = 0.001$) does not change the results noticeably. Similar remarks apply to the case when $\epsilon_I = 0$.

In order to run `COSMOMC` for a theoretical model we not only need prior ranges we also need a covariance matrix which is used in Markov Chain Monte Carlo sampling. Covariance matrices are also provided with `COSMOMC` for a large number of theoretical models. However, for a new model we must find the covariance matrix by running `COSMOMC` for a few test cases and we have also done that.

In the diagonal panels of figure (2) we show (for the case $\epsilon_I = 0.001$) the marginal posterior probability distributions of the parameters T, δN, n_s and A_s which characterize the primordial power spectrum, and in the other panels we show the 68% and 95% confidence regions. We do not show plots for rest of the cosmological parameters (which have expected behavior) since we are primarily interested in the parameters of the primordial power spectrum. From the figure we can see that δN can be well constrained using the combined WMAP nine year and Planck data, while we can put only an upper limit on T.

In table 2 we present a summary of the cosmological parameters for the standard power law model and pre-inflationary model for the joint WMAP nine year and Planck data for the two cases $\epsilon_I = 0.001$ and $\epsilon_I = 0$. Apart from showing the best fit values of the cosmological parameters for the power law and the pre-inflationary model, we also give the 68% limit for the parameters. As we noted above, though we can constrain δN, we can only give an upper bound on T. There is not much change in the values of the standard cosmological parameters and their errors when we replace the standard power law model with the pre-inflationary model, while there is a slight improvement in the $-\log$ likelihood by 1.945. However this improvement cannot be considered significant, in particular when the error bars on the extra parameters δN and T are large. In figure (3), we plot the TT angular
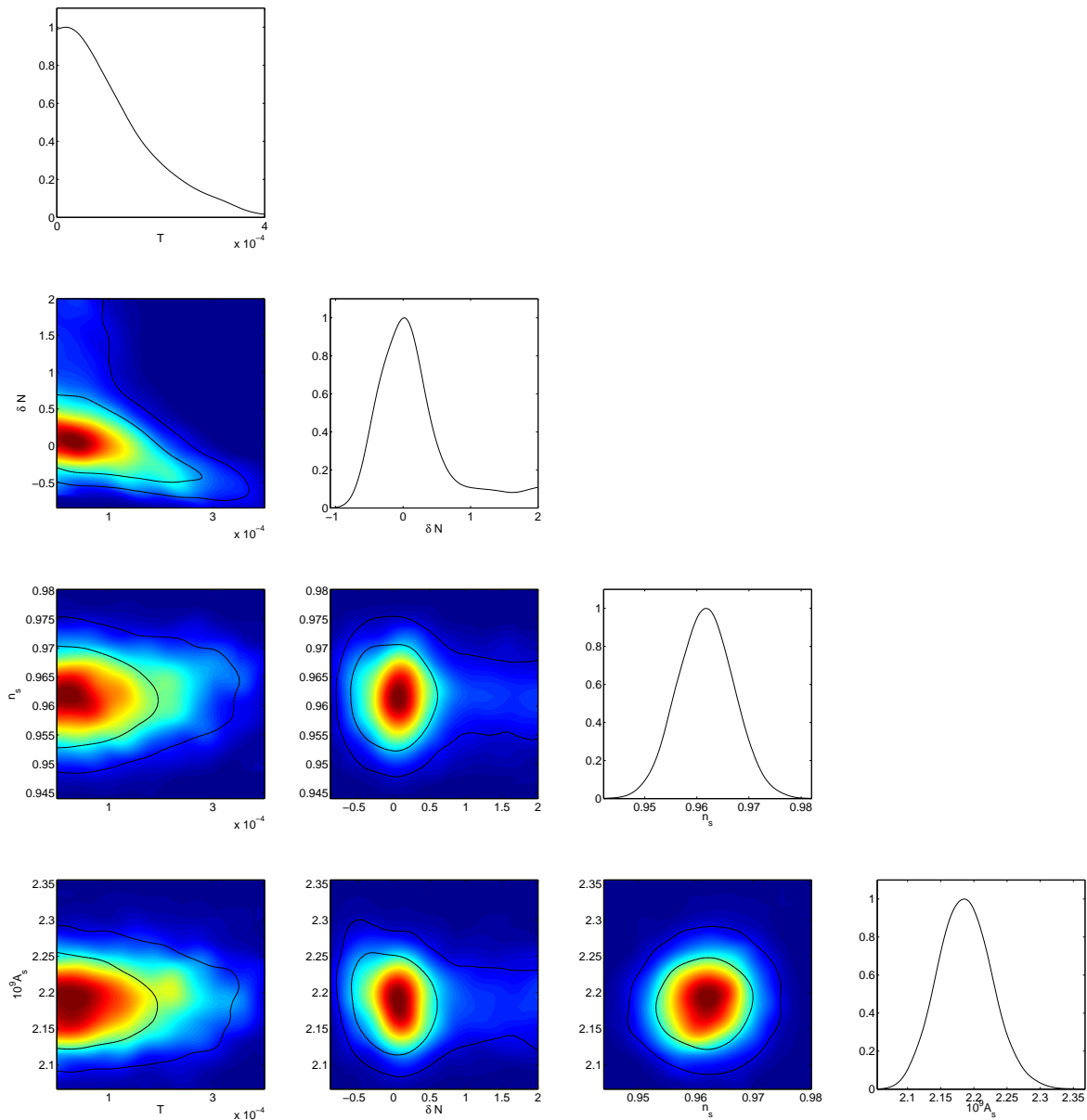
Figure 2. The diagonal panels in this figure show the one dimensional marginalized probability distributions and the other panels show the 68% and 95% confidence regions of the parameters for the WMAP nine year+ Planck data for the pre-inflationary model for $\epsilon_I = 0.001$. Since the rest of the cosmological parameters show the expected behavior we do not show plots for those.

power spectrum of CMB corresponding to the pre-inflationary model with $\epsilon_I = 0.001$, and the power spectrum for the standard power law case, using the best fit values of parameters (and the upper bound on T) from table 2, with the WMAP nine year and Planck datasets.

	Power Law		Pre-Inflation ($\epsilon_I = 0.001$)		Pre-Inflation ($\epsilon_I = 0$)	
Parameter	Best Fit	68% Limit	Best Fit	68%Limit	Best Fit	68 % Limit
$\Omega_b h^2$	0.02238	0.02235 ± 0.00020	0.02234	0.02235 ± 0.00019	0.02226	0.02235 ± 0.00019
$\Omega_c h^2$	0.1159	0.1165 ± 0.0020	0.1167	0.1165 ± 0.0021	0.1171	0.1165 ± 0.0021
Ω_Λ	0.708	0.704 ± 0.012	0.702	0.704 ± 0.012	0.700	0.704 ± 0.012
$10^9 A_s$	2.186	2.186 ± 0.038	2.178	2.186 ± 0.040	2.163	2.187 ± 0.039
n_s	0.962	0.962 ± 0.005	0.961	0.961 ± 0.005	0.960	0.962 ± 0.005
τ	0.089	0.089 ± 0.009	0.087	0.089 ± 0.009	0.083	0.089 ± 0.010
H_0	69.04	68.74 ± 0.97	68.58	68.73 ± 0.96	68.40	68.75 ± 0.98
δN			0.081	0.160 ± 0.549	0.017	0.171 ± 0.552
T				$< 1.2867 \times 10^{-4}$		$< 1.2782 \times 10^{-4}$
$-\log \mathcal{L}$	8691.8200			8689.8750		8689.9980

Table 2. Best fit cosmological parameters estimated from the WMAP nine year + Planck data for the power law model, and the pre-inflationary model with $\epsilon = 0.001$ and 0 with δN and T as extra parameters. (For T we only obtain upper bounds.) Because of skewed non-gaussian distributions the values in the 3rd, 5th and 7th columns are not centred about the best fit values.
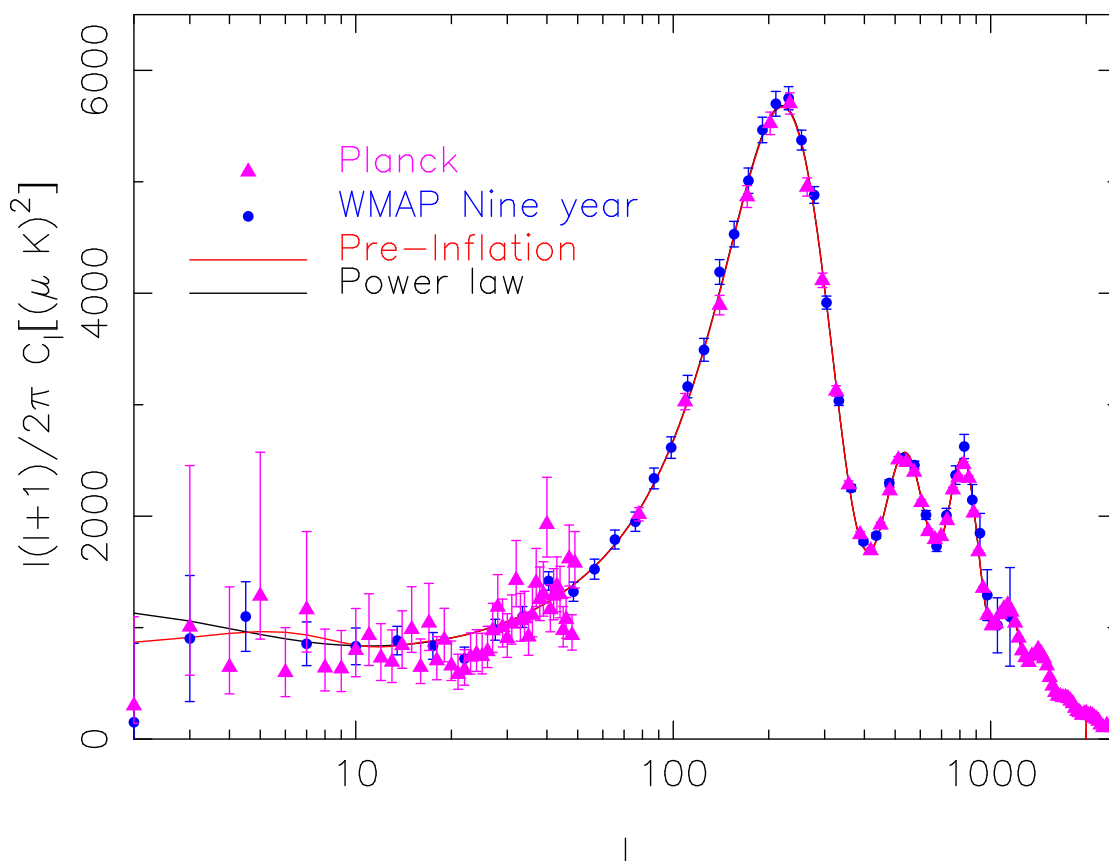
Figure 3. We show the C_l^{TT} for Planck (pink filled triangles) and the WMAP nine year data (large blue dots). The black and red curves respectively show the best-fit theoretical C_l^{TT} for the standard power law model and the pre-inflationary model with $\epsilon_I = 0.001$ using values from table 2. For the latter curve, $\delta N = 0.081$ and $T = 1.2867 \times 10^{-4}$ Mpc^{-1}.

5 Discussion and conclusions

Considering a radiation dominated era prior to inflation we have found the primordial power spectrum of adiabatic scalar perturbations as given by eq. (3.65) and shown in figure 1. In the previous literature where a pre-inflationary radiation era has been discussed, either a modification of the mode functions (due to the presence of a prior radiation era) is considered [6–13], or, a thermal distribution of inflaton quanta (which carries a signature of their prior thermal equilibrium) has been considered [46]. While the first tends to lower the quadrupole moment of the CMB TT anisotropy spectrum, the latter enhances the power at large angular scales. We have included in this study both these phenomena as a consistent approach to determine the physics of the subsequent inflationary era. In figure 1 where we plot the Primordial Power Spectrum with the $|C_1 - C_2|^2$ and the coth term one sees that the damping of the power on large scales due to $|C_1 - C_2|^2$ overwhelms the enhancement due to the coth(k/(2T)] factor. (The ringing behaviour in the power spectrum seen in figure 1 is associated with the abrupt transition from the radiation to the inflationary era at t_i — though a and H are continuous at t_i, ϵ is not. This behaviour is also seen in figure 1 of ref. [6] and has been discussed in ref. [13]. In refs. [78, 79] particle creation associated with the abrupt transition in ϵ between eras has been regulated by letting the Bogolyubov parameter $\beta_k \to \beta_k e^{-\bar{\tau}k}$ for some very small but finite time duration $\bar{\tau}$ of the transition. If one tries to consider ϵ varying over a short time scale during the transition between eras then it will not be possible to obtain analytical solutions for the equations for the mode functions.)

Using WMAP nine year and Planck data, we find that the upper bound on the comoving temperature of inflaton is given by $T \lesssim 1.28 \times 10^{-4}\mathrm{Mpc}^{-1}$, and the best fit value of the minimum number of e-folds is 0.081 more than the minimum value required to solve the horizon and flatness problems (see table 2 and figure 2). These results do not change significantly for the smaller energy scale of inflation ($\epsilon_I = 0$). We have thus improved the existing constraint on the comoving temperature of the inflaton particles from $T \lesssim 10^{-3}\mathrm{Mpc}^{-1}$ (according to ref. [46]) to $T \lesssim 10^{-4}\mathrm{Mpc}^{-1}$, i.e. by one order of magnitude.[4]

Our analysis shows that considering the effect of both the modified mode functions and the temperature of the inflaton quanta on the primordial power spectrum marginally improves the likelihood compared to the standard case though, as mentioned earlier, its significance is diminished because of the large error bars for δN and T. We also depict the CMB power spectrum in the presence of a pre-inflationary radiation era with the WMAP nine year and Planck data in figure 3. It is evident from the plot that the best fit spectrum tends to lower the quadrupole moment. However the decrease is not significant enough for us to claim that the low l anomaly present in the Planck data can indicate a pre-inflationary radiation era.

One motivation of having a pre-inflationary radiation era is to get rid of the infrared divergences which appear in the field theoretic treatment of perturbations in the inflationary scenario. We have shown that the corrected power spectrum obtained in our analysis is still infrared safe, even with the coth term, and hence curing infrared divergences with the existence a pre-inflationary era holds good in our scenario as well. We have also discussed the conditions under which the radiation dominated era prior to inflation can be treated in the fluid approximation, as needed in the standard cosmological perturbation theory.

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[4] If we repeat the analysis of ref. [46], i.e. without modifications of the mode functions, and include only one extra parameter T in COSMOMC then, with the newer WMAP9 + Planck data, we again find $T \lesssim 10^{-4}\mathrm{Mpc}^{-1}$. This will imply an increase of 2 in the minimum δN of $7 - 32$ for GUT-electroweak scale inflation obtained in ref. [46]. (δN is large in this analysis because there is no suppression of the coth factor by the $|C_1 - C_2|^2$ factor.)

Acknowledgments

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